# SEGRE EMBEDDINGS AND THE CANONICAL IMAGE OF A CURVE

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ABSTRACT. We prove that there is no $g$ for which the canonical embedding of a general curve of genus $g$ lies on the Segre embedding of any product of three or more projective spaces.

## 1. INTRODUCTION

If $g$ is composite then the canonical embedding of a general curve of genus $g$ lies on the Segre embedding of a product of two projective spaces. For example, a general curve of genus $4$ lies on the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ while a general curve of genus $6$ lies on the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^2$. These facts have applications concerning the structure Chow ring of $\mathcal{M}_g$ as illustrated in [Fab, p. 421] and [Pen, p. 26]. The aim of this note is to prove that, by contrast, there is no $g$ for which the canonical embedding of a general curve of genus $g$ lies on the Segre embedding of any product of three or more projective spaces.

To prove our main result we first give the following criterion for a general curve to lie on some Segre embedding $\mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_n} \to \mathbf{P}^{g-1}$.

**Proposition 1.1.** *The canonical image of a general curve $C$ of genus $g$ lies on some Segre embedding $\mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_n} \to \mathbf{P}^{g-1}$ if and only if $C$ admits line bundles $L_1, \ldots, L_n$ such that $\bigotimes_{i=1}^{n} L_i = K_C$, $h^0(C, L_i) = r_i + 1$ and $\prod_{i=1}^{n}(r_i+1) = g$.*

We then prove the following stronger result of independent interest.

**Theorem 1.2.** *Let $C$ be a general curve of genus $g$ and let $n \geqslant 3$. If $C$ admits line bundles $L_1, \ldots, L_n$ such that $\sum_{i=1}^{n} \deg L_i = 2g-2$ and $h^0(C, L_i) = r_i + 1 \geqslant 2$ for $i = 1, \ldots, n$, then $n = 3$ and*

$$\prod_{i=1}^{3}(r_i+1) < \left(\prod_{i=1}^{3}(r_i+1)\right)\left(\frac{r_1+r_2+r_3+2}{r_1+r_2+r_3+2-r_1r_2r_3}\right) \leqslant g.$$

*Moreover, up to permutation of the $r_i$, we have*

(a) *$r_1 = r_2 = 1$ and $1 \leqslant r_3 \leqslant \frac{g}{4} - 2$ or*
(b) *$r_1 = 1$, $r_2 = 2$ and $r_3 = 2, 3,$ or $4$.*

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A more conceptual interpretation of the theorem is the following corollary.

**Corollary 1.3.** *Let $C$ be a general curve of genus $g$.*

(a) *If $g$ is composite then the canonical image of $C$ lies on some Segre embedding*

$$\mathbf{P}^{r_1} \times \mathbf{P}^{r_2} \to \mathbf{P}^{g-1}.$$

(b) *If $n \geqslant 3$ then the canonical image of $C$ does not lie on any Segre embedding*

$$\mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_n} \to \mathbf{P}^{g-1}.$$

Statement (a) in Corollary 1.3 is well-known to experts and follows from Brill-Noether theory. We include a proof for completeness. We refer to [ACGH, Chap. IV] and [ACG, Chap. XXI] for the basic results in Brill-Noether theory and [ACGH, p. xv] for notation. For the non-expert we give a quick exposition of some of the main results of the theory in Section 2. Throughout we assume that $g$ is an integer greater than or equal to $3$.

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## 2. Brill-Noether theory

For all smooth curves $C$ of genus $g$ there exist moduli schemes $W^r_d(C)$ whose closed points consist of equivalence classes of degree $d$ line bundles on $C$ with at least $r+1$ global sections. Explicitly the closed points of $W^r_d(C)$ are identified with the set

$$\{L \mid L \in \operatorname{Pic}(C), \deg L = d \text{ and } h^0(C, L) \geqslant r+1\}.$$

For the construction of $W^r_d(C)$ see [FL, p. 279] or [ACGH, p. 176].

Brill-Noether theory studies the geometry of such $W^r_d(C)$. Central to the theory is the Brill-Noether number and the Petri-map. The Brill-Noether number is defined by

$$\rho(g, r, d) := g - (r+1)(g + r - d).$$

See [ACGH, p. 159] for an explanation as to how this number arises. On the other hand, the Petri-map is defined for all line bundles $L$ on $C$. If $K_C$ denotes the canonical bundle of $C$ then the Petri-map is the cup-product

$$\mu(L) : \mathrm{H}^0(C, L) \otimes \mathrm{H}^0(C, L^\vee \otimes K_C) \to \mathrm{H}^0(C, K_C).$$

Both the Brill-Noether number and the Petri-map have remarkable geometric implications as seen in the two main theorems of Brill-Noether theory which we now describe. The first theorem applies to all smooth curves and is the result of work by Kempf, Kleiman-Laksov and Fulton-Lazarsfeld.

**Theorem 2.1** (Brill-Noether theory theorem I [KL], [FL])**.** *Let $C$ be a smooth curve of genus $g$.*

(a) *If $\rho(g, r, d) \geqslant 0$ then $W^r_d(C) \neq \varnothing$ and every irreducible component of $W^r_d(C)$ has dimension greater than or equal to $\rho(g, r, d)$.*
(b) *If $\rho(g, r, d) \geqslant 1$ then $W^r_d(C)$ is connected.*

If $C$ is a general curve then the converse to Theorem 2.1 (a) holds – we state this in the second main theorem (Theorem 2.4). That the converse to Theorem 2.1 (a) holds for general curves was first proved by Griffiths-Harris [GH] using a degeneration argument.

Petri conjectured that $\mu(L)$ is injective for all line bundles $L$ on a general curve $C$. (This is implicit in [Pet]. See the footnote on [ACGH, p. 215] for a discussion.) Arbarello-Cornalba clarified the geometric implications of this conjecture.

**Theorem 2.2** (Arbarello-Cornalba theorem [AC, Theorem 0.3])**.** *Let $C$ be a general curve of genus $g$. If $\mu(L)$ is injective for all line bundles $L$ on $C$ and $W^r_d(C) \neq \varnothing$ then every irreducible component of $W^r_d(C)$ has dimension $\rho(g, r, d)$ and $W^r_d(C)$ is non-singular away from $W^{r+1}_d(C)$.*

In [Gie] Gieseker used a degeneration argument, and which was subsequently streamlined by Eisenbud-Harris [EH], to prove that $\mu(L)$ is injective for all line bundles on a general curve. Lazarsfeld, without using degenerations, also gave an independent proof [Laz, p. 299]. The fact that $\mu(L)$ is injective for all line bundles on a general curve is sometimes referred to as the Gieseker-Petri theorem.

**Theorem 2.3** (Gieseker-Petri theorem [Gie, Theorem 1.1, p. 251])**.** *If $C$ is a general curve of genus $g$ then the cup-product $\mu(L)$ is injective for all line bundles $L$ on $C$.*

The above discussion, combined with [FL, Corollary 2.4, p. 280], is summarized in the second main theorem of Brill-Noether theory.

**Theorem 2.4** (Brill-Noether theory theorem II [AC], [FL], [Gie],[GH], [Laz])**.** *Let $C$ be a general curve of genus $g$.*

(a) *If $W^r_d(C) \neq \varnothing$ then $\rho(g, r, d) \geqslant 0$, $W^r_d(C)$ is of pure dimension $\rho(g, r, d)$ and $W^r_d(C)$ is non-singular away from $W^{r+1}_d(C)$.*
(b) *If $\rho(g, r, d) \geqslant 1$ then $W^r_d(C)$ is irreducible.*

The results of this note depend on Theorem 2.1 (a), Theorem 2.3, and Theorem 2.4 (a).

## 3. THE $n$-FOLD PETRI-MAP AND THE PROOF OF PROPOSITION 1.1

Proposition 1.1 relies on the following lemma which implies that, for a general curve, the $n$-fold Petri-map is injective.

**Lemma 3.1.** *Let $C$ be a general curve of genus $g$. For all line bundles $L_1, \ldots, L_n$ on $C$ such that $\bigotimes_{i=1}^{n} L_i \cong K_C$ the cup-product $\bigotimes_{i=1}^{n} \mathrm{H}^0(C, L_i) \to \mathrm{H}^0(C, K_C)$ is injective.*

*Proof.* The case $n = 1$ is trivial. The case $n = 2$ is the Gieseker-Petri theorem (Theorem 2.3). Let $n \geqslant 3$. Without loss of generality we may assume that $\mathrm{H}^0(C, L_i) \neq 0$, $i = 1, \ldots n$.

The cup-product factors

$$\begin{tikzcd} \mathrm{H}^0(C,L_1)\otimes\cdots\otimes\mathrm{H}^0(C,L_n) \arrow[rr] \arrow[d] & & \mathrm{H}^0(C,K_C)\ . \\ \mathrm{H}^0(C,L_1\otimes L_2)\otimes\mathrm{H}^0(C,L_3)\otimes\cdots\otimes\mathrm{H}^0(C,L_n) \arrow[urr] & & \end{tikzcd}$$

By induction the diagonal arrow is injective. It thus suffices to show that the downward arrow is injective. For this we reduce to showing that the cup-product

$$\Phi : \mathrm{H}^0(C, L_1) \otimes \mathrm{H}^0(C, L_2) \to \mathrm{H}^0(C, L_1 \otimes L_2)$$

is injective.

By assumption there exist non-zero sections $\sigma_i \in \mathrm{H}^0(C, L_i), i = 3, \dots n$. These produce (via cup-product) a non-zero section

$$\sigma = \sigma_3 \cdots \sigma_n \in \mathrm{H}^0(C, L_3 \otimes \cdots \otimes L_n).$$

Since $\sigma \neq 0$ multiplication by $\sigma$ yields the following two injections

$$\mathrm{H}^0(C, L_2) \to \mathrm{H}^0(C, L_2 \otimes \cdots \otimes L_n)$$

and

$$\mathrm{H}^0(C, L_1 \otimes L_2) \to \mathrm{H}^0(C, K_C).$$

Using the above we produce (by cup-product) the commutative diagram

$$\begin{tikzcd} \mathrm{H}^0(C,L_1)\otimes\mathrm{H}^0(C,L_2) \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}\otimes\cdot\sigma"'] & \mathrm{H}^0(C,L_1\otimes L_2)\ . \arrow[d, "\cdot\sigma"] \\ \mathrm{H}^0(C,L_1)\otimes\mathrm{H}^0(C,L_2\otimes\cdots\otimes L_n) \arrow[r, "\mu(L_1)"] & \mathrm{H}^0(C,K_C) \end{tikzcd}$$

The vertical arrows of the above diagram are injective, as just noted, whereas bottom arrow of the diagram is injective by the Gieseker-Petri theorem. Hence the top arrow $\Phi$ is injective. □

We now use Lemma 3.1 to prove Proposition 1.1.

*Proof of Proposition 1.1.* Let $\eta : C \to \mathbf{P}^{g-1}$ be the canonical map. If $C$ is contained in the image of some Segre embedding $\phi : \mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_n} \to \mathbf{P}^{g-1}$ then there exists a closed immersion $\psi : C \to \mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_n}$ making the diagram

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[d, "\psi"'] & \mathbf{P}^{g-1} \\ \mathbf{P}^{r_1}\times\cdots\times\mathbf{P}^{r_n} \arrow[ur, "\phi"'] & \end{tikzcd}$$

commute.

For every $1 \leqslant i \leqslant n$, let $\pi_i$ denote the projection of $\mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_n}$ onto the $i$-th factor and set $L_i := (\pi_i \circ \psi)^* \mathcal{O}_{\mathbf{P}^{r_i}}(1) \cong \psi^*(\pi_i^* \mathcal{O}_{\mathbf{P}^{r_i}}(1))$. We then obtain

$$K_C \cong \eta^* \mathcal{O}_{\mathbf{P}^{g-1}}(1) \cong (\phi \circ \psi)^* \mathcal{O}_{\mathbf{P}^{g-1}}(1) \cong L_1 \otimes \cdots \otimes L_n.$$

Since the canonical image of $C$ is non-degenerate we conclude that $h^0(C, L_i) \geqslant r_i + 1$ for $i = 1, \ldots, n$.

Finally, since $\eta$ is induced by the complete canonical series, we conclude that the cup-product $\bigotimes_{i=1}^{n} \mathrm{H}^0(C, L_i) \to \mathrm{H}^0(C, K_C)$ is surjective. By Lemma 3.1 the cup-product is injective and, by assumption, $\prod_{i=1}^{n}(r_i + 1) = g$. It follows that $h^0(C, L_i) = r_i + 1$ for $i = 1, \ldots, n$ .

Conversely, given such $L_1, \ldots, L_n$ we get regular maps $C \to \mathbf{P}^{r_i}$ for $i = 1, \ldots, n$. We thus can make a regular map $\eta : C \to \mathbf{P}^{g-1}$ (induced by a (sub)-canonical series) by composition $C \longrightarrow \mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_n} \xrightarrow{\text{Segre}} \mathbf{P}^{g-1}$ . By Lemma 3.1 the cup-product $\bigotimes_{i=1}^{n} \mathrm{H}^0(C, L_i) \to \mathrm{H}^0(C, K_C)$ is injective. Since $\prod_{i=1}^{n}(r_i + 1) = g$ it is also surjective. We thus conclude that the resulting map $\eta$ is given by the complete canonical series. $\square$

## 4. Proof of main Theorem and its corollary

We first prove Corollary 1.3 (a). We then prove Theorem 1.2 from which we deduce Corollary 1.3 (b).

**The case $n = 2$ and $g$ is composite.** When $n = 2$ and $g$ is composite it is easy to prove that, in its canonical embedding, a general curve of genus $g$ lies on the image of some (non-trivial) Segre embedding $\mathbf{P}^{r_1} \times \mathbf{P}^{r_2} \to \mathbf{P}^{g-1}$.

*Proof of Corollary 1.3 (a).* Since $g$ is composite we can write $g = (r_1 + 1)(r_2 + 1)$ with $r_i \geqslant 1$. Set $d_1 = \frac{r_1 g}{r_1 + 1} + r_1 = r_1 r_2 + 2r_1$. Then $\rho(g, r_1, d_1) = 0$ so, by Theorem 2.1 (a), $C$ admits a line bundle $L_1$ with at least $r_1 + 1$ global sections. On the other hand $\rho(g, r_1 + 1, d_1) < 0$, so Theorem 2.4 (a) implies that $W^{r_1+1}_{d_1} = \varnothing$. Hence $L_1$ has exactly $r_1 + 1$ global sections. Set $L_2 := L_1^{\vee} \otimes K_C$. Then, by the Riemann-Roch theorem, we obtain $h^0(C, L_2) = g - d_1 + r_1$. Simplifying and using our expressions for $g$ and $d_1$ above we obtain
$h^0(C, L_2) = (r_1 + 1)(r_2 + 1) - r_1 r_2 - 2r_1 + r_1 = r_2 + 1$. The assertion now follows from Proposition 1.1. $\square$

**The case $n \geqslant 3$.** When $n \geqslant 3$ and $r_i \geqslant 1$, for $i = 1, \ldots, n$, the situation is in stark contrast to the case $n = 2$. Indeed we prove that there is no $g$ for which the canonical embedding of a general curve lies on the Segre embedding of any product of three or more projective spaces. We deduce this result from Theorem 1.2 whose proof occupies the rest of this section. The theorem follows from the following more general observation.

**Proposition 4.1.** *Let $C$ be a general curve of genus $g$. Let $d$ be a non-negative integer and let $q := \left\lfloor \frac{d}{g} \right\rfloor$. If $C$ admits line bundles $L_1, \ldots, L_n$ such that $r_i+1 = h^0(C, L_i) \geqslant 2$ and $\deg \bigotimes_{i=1}^n L_i = d$, then $n < 2q+2$ and $g \geqslant \frac{1+\left(\sum_{i=1}^n r_i\right)}{-n+(q+1)+\left(\sum_{i=1}^n \frac{1}{r_i+1}\right)}$.*

*Proof.* Since $C$ is general, if such $L_i$ exist then, by Theorem 2.4 (a),

$$\rho(g, r_i, d_i) = g - (r_i+1)(g+r_i-d_i) \geqslant 0 \text{ for } i = 1, \ldots, n.$$

Solving for $d_i$ we conclude $d_i \geqslant g + r_i - \frac{g}{r_i+1}$ for $i = 1, \ldots, n$. Let $r$ denote the remainder obtained by dividing $d$ by $g$. Then $0 \leqslant r < g$ and

$$d = (q+1)g + r - g = \sum_{i=1}^n d_i \geqslant ng + \left(\sum_{i=1}^n r_i\right) - \left(\sum_{i=1}^n \frac{g}{r_i+1}\right).$$

Rearranging we obtain

$$\left(n - (q+1) - \left(\sum_{i=1}^n \frac{1}{r_i+1}\right)\right) g \leqslant r - g - \left(\sum_{i=1}^n r_i\right) < 0. \tag{4.1}$$

Since $g > 0$ we conclude

$$n - (q+1) - \left(\sum_{i=1}^n \frac{1}{r_i+1}\right) < 0. \tag{4.2}$$

Now by assumption $r_i \geqslant 1$, for $i = 1, \ldots, n$. Thus

$$\sum_{i=1}^n \frac{1}{r_i+1} \leqslant \frac{n}{2}. \tag{4.3}$$

Using equations (4.3) and (4.2) we deduce that $n < 2q+2$. Finally, dividing equation (4.1) by equation (4.2) we obtain

$$g \geqslant \frac{-r+g+\left(\sum_{i=1}^n r_i\right)}{-n+(q+1)+\left(\sum_{i=1}^n \frac{1}{r_i+1}\right)} \geqslant \frac{1+\left(\sum_{i=1}^n r_i\right)}{-n+(q+1)+\left(\sum_{i=1}^n \frac{1}{r_i+1}\right)}. \tag{4.4}$$

□

We now use Proposition 4.1 to prove Theorem 1.2.

*Proof of Theorem 1.2.* Set $d_i = \deg L_i$, $i = 1, \ldots, n$ and set $d = 2g-2$. Then $q := \left\lfloor \frac{d}{g} \right\rfloor = 1$ and the remainder $r$ equals $g-2$. Since $r_i \geqslant 1$, for $i = 1, \ldots, n$, applying Proposition 4.1 we

conclude that $n < 4$. Since $n \geqslant 3$ we conclude that $n = 3$. Substituting $n = 3$ and $r = g - 2$ into equation (4.4) we obtain

$$g \geqslant \frac{2 + \sum_{i=1}^{3} r_i}{-1 + \sum_{i=1}^{3} \frac{1}{r_i+1}}.$$

Rearranging we obtain

$$g \geqslant \left(\prod_{i=1}^{3}(r_i+1)\right)\left(\frac{r_1+r_2+r_3+2}{r_1+r_2+r_3+2-r_1r_2r_3}\right) > \prod_{i=1}^{3}(r_i+1).$$

Since $1 < \sum_{i=1}^{3} \frac{1}{r_i+1}$ we conclude, up to permutation of the $r_i$, that $r_1 = r_2 = 1$, $r_3 \geqslant 1$ or $r_1 = 1, r_2 = 2$ and $2 \leqslant r_3 \leqslant 4$. Finally if $r_1 = r_2 = 1$ and $r_3 \geqslant 1$ then the condition $\prod_{i=1}^{3}(r_i+1) < g$ implies that $1 \leqslant r_3 \leqslant \frac{g}{4} - 2$. □

Having proved Theorem 1.2, we now deduce Corollary 1.3 (b).

*Proof of Corollary 1.3 (b).* Let $C$ be a general curve of genus $g$. Suppose that $C$ lies on the image of some Segre embedding $\mathbf{P}^{r_1} \times \cdots \times \mathbf{P}^{r_n} \to \mathbf{P}^{g-1}$. By Proposition 1.1, $C$ admits line bundles $L_1, \ldots, L_n$ such that $\bigotimes_{i=1}^{g} L_i = K_C$, $h^0(C, L_i) = r_i + 1$ and $\prod_{i=1}^{n}(r_i+1) = g$. By Theorem 1.2, $n = 3$ and $g > \prod_{i=1}^{3}(r_i+1)$. This is a contradiction. □

## References


[ACG] Arbarello, E., Cornalba, M. and Griffiths, P. A. *Geometry of algebraic curves. Vol. II* (Springer-Verlag, Berlin Heidelberg, 2011).

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P. A. and Harris, J., *Geometry of algebraic curves. Vol. I* (Springer-Verlag, New York, 1985).

[AC] Arbarello, E. and Cornalba, M., *On a conjecture of Petri*, Comment. Math. Helv. **56** (1981) 1–38.

[EH] Eisenbud, D. and Harris, J., *A simpler proof of the Gieseker-Petri theorem on special divisors*, Invent. Math. **74** (1983) 269–280.

[Fab] Faber, C., *Chow rings of moduli spaces of curves. II. Some results on the Chow ring of* $\overline{\mathcal{M}_4}$ Ann. of Math. (2) (1990) 421–449.

[FL] Fulton, W. and Lazarsfeld, R., *On the connectedness of degeneracy loci and special divisors*, Acta Math. **146** (1981) 271–283.

[Gie] Gieseker, D., *Stable curves and special divisors: Petri's conjecture*, Invent. Math. **66** (1982) 251–275.

[GH] Griffiths, P. and Harris, J., *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. **47** (1980) 233–272.

[KL] Kleiman, S. L. and Laksov, D., *On the existence of special divisors*, Amer. J. Math. **94** (1972) 431–436.

[Laz] Lazarsfeld, R., *Brill–Noether–Petri without degenerations*, J. Differential Geom. **23** (1986) 299–307.

[Pen] Penev, N. *The Chow ring of* $\mathcal{M}_6$, PhD thesis, Standford University (2009) pp.46.

[Pet] Petri, K. *Über Spezialkurven. I*, Math. Ann. **93** (1925) 182–209.

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